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A note on the Pfaffian integration theorem

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Abstract
Two alternative, fairly compact proofs are presented of the Pfaffian integration theorem that surfaced in the recent studies of spectral properties of Ginibre's Orthogonal Ensemble. The first proof is based on a concept of the Fredholm Pfaffian; the second proof is purely linear algebraic.

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1. Introduction

In the recent studies of spectral properties of Ginibre's orthogonal ensemble (Ginibre 1965) of real asymmetric random matrices[3], the following theorem was presented by Kanzieper and Akemann (2005) and Akemann and Kanzieper (2007):

Pfaffian integration theorem. *Let $\pi(\mathrm{d}z)$ be an arbitrary measure on $z \in \mathbb{C}$ with finite moments. Define the function $Q_n(z,w) = \underline{\boldsymbol{q}}(z)\boldsymbol{\mu}\underline{\boldsymbol{q}}^{\mathrm{T}}(w)$ in terms of the vector $\underline{\boldsymbol{q}}(z) = (q_0(z),\ldots,q_{n-1}(z))$ composed of arbitrary polynomials $q_j(z)$ of the jth order, and of an $n\times n$ antisymmetric matrix $\boldsymbol{\mu}$. Then*

$$\frac{1}{\ell!}\prod_{j=1}^{\ell}\int_{\mathbb{C}}\pi(\mathrm{d}z_j)\,\mathrm{pf}\begin{bmatrix} Q_n(z_i,z_j) & Q_n(z_i,\bar z_j)\\ Q_n(\bar z_i,z_j) & Q_n(\bar z_i,\bar z_j)\end{bmatrix}_{1\leqslant i,j\leqslant \ell} = e_\ell\left(\frac{1}{2}\mathrm{tr}(\boldsymbol{\upsilon}^1),\ldots,\frac{1}{2}\mathrm{tr}(\boldsymbol{\upsilon}^\ell)\right),$$

where $e_\ell(p_1,\ldots,p_\ell)$ are the elementary symmetric functions[4] written as polynomials of the power sums, and $n\times n$ matrix $\boldsymbol{\upsilon}$ is $\boldsymbol{\upsilon} = \boldsymbol{\mu}\boldsymbol{g}$ with

$$\boldsymbol{g} = \int_{\mathbb{C}}\pi(\mathrm{d}z)(\underline{\boldsymbol{q}}^{\mathrm{T}}(\bar z)\underline{\boldsymbol{q}}(z) - \underline{\boldsymbol{q}}^{\mathrm{T}}(z)\underline{\boldsymbol{q}}(\bar z)).$$

[3] For a discussion of physical applications of Ginibre's random matrices, the reader is referred to the detailed paper by Akemann and Kanzieper (2007).
[4] Up to a factorial prefactor, the elementary symmetric functions $e_\ell(p_1,\ldots,p_\ell)$ coincide with the zonal polynomials $Z_{(1^\ell)}(p_1,\ldots,p_\ell) = \ell!e_\ell(p_1,\ldots,p_\ell)$ appearing in the original formulation of the theorem.

This theorem has been a key ingredient of the recent calculation (Kanzieper and Akemann 2005, Akemann and Kanzieper 2007) of the probability $p_{n,k}$ to find exactly k real eigenvalues in the spectra of $n \times n$ real asymmetric random matrices drawn from Ginibre's orthogonal ensemble. An earlier attempt to address the same problem is due to Edelman (1997).

Remark 1.1. The explicit form of $e_\ell(p_1, \ldots, p_\ell)$ is well known (see, e.g., Macdonald 1998):

$$e_\ell(p_1, \ldots, p_\ell) = (-1)^\ell \sum_{|\boldsymbol{\lambda}|=\ell} \prod_{j=1}^{g} \frac{1}{\sigma_j!} \left(-\frac{p_{\ell_j}}{\ell_j} \right)^{\sigma_j}. \tag{1.1}$$

The notation $\boldsymbol{\lambda} = \left(\ell_1^{\sigma_1}, \ldots, \ell_g^{\sigma_g}\right)$ stands for the frequency representation of the partition $\boldsymbol{\lambda}$ of the size $|\boldsymbol{\lambda}| = \ell$. It implies that the part ℓ_j appears σ_j times so that $\ell = \sum_{j=1}^{g} \ell_j \sigma_j$, where g is the number of nonzero parts of the partition.

An immediate corollary of equation (1.1) is the identity

$$\sum_{\ell=0}^{\infty} \tau^\ell e_\ell(p_1, \ldots, p_\ell) = \exp\left(\sum_{j=1}^{\infty} (-1)^{j-1} \tau^j \frac{p_j}{j} \right). \tag{1.2}$$

Remark 1.2. The Pfaffian integration theorem can be viewed as a generalization of the Dyson integration theorem (Dyson 1970, Mahoux and Mehta 1991) for the case where the quaternion kernel $\mathcal{Q}_n(z, w)$ represented by the 2×2 matrix[5]

$$\Theta[\mathcal{Q}_n(z, w)] = \tilde{\boldsymbol{J}}^{-1} \begin{pmatrix} Q_n(z, w) & Q_n(z, \bar{w}) \\ Q_n(\bar{z}, w) & Q_n(\bar{z}, \bar{w}) \end{pmatrix}$$

does *not* satisfy the projection property

$$\int_{\mathbb{C}} \mathrm{d}\pi(w) \mathcal{Q}_n(z, w) \mathcal{Q}_n(w, z') = \mathcal{Q}_n(z, z').$$

The original proof (Akemann and Kanzieper 2007) of the theorem involved an intricate topological interpretation of the ordered Pfaffian expansion combined with the term-by-term integration that spanned dozens of pages. In the present contribution, we provide two alternative, concise proofs of slight variations of the Pfaffian integration theorem. They are formulated in the form of the two theorems and represent the main result of our note.

Theorem 1. *Let (X, m) be a measure space, and the vectors $\underline{\varphi}^+(x)$ and $\underline{\varphi}^-(x)$ be composed of measurable functions $\underline{\varphi}^\pm(x) = \left(\varphi_0^\pm(x), \ldots, \varphi_{n-1}^\pm(x)\right)$ from X to $\mathbb{C}$. Define the functions*

$$\Phi_n^{\pm\pm}(x, y) = \underline{\varphi}^\pm(x) \boldsymbol{\mu} \underline{\varphi}^{\pm\mathrm{T}}(y),$$

where $\boldsymbol{\mu}$ is an $n \times n$ antisymmetric matrix. Then[6]

$$\frac{1}{\ell!} \prod_{j=1}^{\ell} \int_X m(\mathrm{d}x_j) \mathrm{pf} \begin{bmatrix} \Phi_n^{++}(x_i, x_j) & \Phi_n^{+-}(x_i, x_j) \\ \Phi_n^{-+}(x_i, x_j) & \Phi_n^{--}(x_i, x_j) \end{bmatrix}_{1 \leqslant i,j \leqslant \ell} = e_\ell \left(\frac{1}{2} \mathrm{tr}(\boldsymbol{v}^1), \ldots, \frac{1}{2} \mathrm{tr}(\boldsymbol{v}^\ell) \right), \tag{1.3}$$

where the $n \times n$ matrix $\boldsymbol{v}$ is $\boldsymbol{v} = \boldsymbol{\mu} \boldsymbol{g}$ with

$$\boldsymbol{g} = \int_X m(\mathrm{d}x) (\underline{\varphi}^{-\mathrm{T}}(x) \underline{\varphi}^+(x) - \underline{\varphi}^{+\mathrm{T}}(x) \underline{\varphi}^-(x)). \tag{1.4}$$

[5] See appendix for the notation.

[6] In what follows, we assume that our measures are such that all integrals are finite.

Theorem 2. *In the notation of theorem 1, assume that the matrix $\boldsymbol{\mu}$ is invertible (hence, n is even). Then*

$$\sum_{\ell=0}^{n/2}\frac{\tau^\ell}{\ell!}\prod_{j=1}^{\ell}\int_X m(\mathrm{d}x_j)\mathrm{pf}\begin{bmatrix}\Phi_n^{++}(x_i,x_j) & \Phi_n^{+-}(x_i,x_j)\\ \Phi_n^{-+}(x_i,x_j) & \Phi_n^{--}(x_i,x_j)\end{bmatrix}_{1\leqslant i,j\leqslant\ell} = \mathrm{pf}\boldsymbol{\mu}\cdot\mathrm{pf}[\boldsymbol{\mu}^{-1\mathrm{T}}-\tau\boldsymbol{g}]. \quad (1.5)$$

Remark 1.3. The equivalence of theorems 1 and 2 is easily established. Indeed, multiplying both sides of equation (1.3) by τ^ℓ and summing up over ℓ from 0 to ∞ with the help of equation (1.2), one finds that the right-hand side turns into

$$\exp\left(\frac{1}{2}\sum_{j=1}^{\infty}(-1)^{j-1}\tau^j\frac{\mathrm{tr}(\boldsymbol{\upsilon}^j)}{j}\right)=\sqrt{\det(\boldsymbol{I}+\tau\boldsymbol{\upsilon})}=\mathrm{pf}\boldsymbol{\mu}\cdot\mathrm{pf}[\boldsymbol{\mu}^{-1\mathrm{T}}-\tau\boldsymbol{g}],$$

given that $\boldsymbol{\mu}$ is invertible. This proves the equivalence for invertible (nondegenerate) $\boldsymbol{\mu}$. Theorem 1 with degenerate $\boldsymbol{\mu}$ of even size follows by the limit transition, and decreasing the size n by 1 is achieved by setting $\varphi_{n-1}^{\pm}\equiv 0$ and nullifying the last, nth, row and column in the matrix $\boldsymbol{\mu}$.

2. Fredholm Pfaffian proof of theorem 1

For $\ell\in\mathbb{Z}^+$, define the infinite sequence

$$\sigma_\ell(n)=\prod_{j=1}^{\ell}\int_X m(\mathrm{d}x_j)\mathrm{pf}\begin{bmatrix}\Phi_n^{++}(x_i,x_j) & \Phi_n^{+-}(x_i,x_j)\\ \Phi_n^{-+}(x_i,x_j) & \Phi_n^{--}(x_i,x_j)\end{bmatrix}_{1\leqslant i,j\leqslant\ell}, \quad (2.1)$$

supplemented by $\sigma_0(n)=1$, and consider the series

$$\mathcal{S}(\tau;n)=\sum_{\ell=0}^{\infty}\frac{\tau^\ell}{\ell!}\sigma_\ell(n). \quad (2.2)$$

By definition (equation (A.3)) introduced by Rains (2000), the function $\mathcal{S}(\tau;n)$ is the Fredholm Pfaffian on the measure space (X,m)

$$\mathcal{S}(\tau;n)=\mathrm{pf}_X[\boldsymbol{J}+\tau\boldsymbol{\Phi}_n]=\sqrt{\det_X[\boldsymbol{I}-\tau\boldsymbol{J}\boldsymbol{\Phi}_n]}. \quad (2.3)$$

(See appendix for the matrix notation used.) Here, $\boldsymbol{\Phi}_n$ is the 2×2 matrix kernel

$$\boldsymbol{\Phi}_n(x,y)=\begin{pmatrix}\Phi_n^{++}(x,y) & \Phi_n^{+-}(x,y)\\ \Phi_n^{-+}(x,y) & \Phi_n^{--}(x,y)\end{pmatrix} \quad (2.4)$$

that can also be written as

$$\boldsymbol{\Phi}_n(x,y)=\begin{pmatrix}\underline{\varphi}^+(x)\underline{\psi}^{+\mathrm{T}}(y) & \underline{\varphi}^+(x)\underline{\psi}^{-\mathrm{T}}(y)\\ \underline{\varphi}^-(x)\underline{\psi}^{+\mathrm{T}}(y) & \underline{\varphi}^-(x)\underline{\psi}^{-\mathrm{T}}(y)\end{pmatrix}=\begin{pmatrix}\underline{\varphi}^+(x)\\ \underline{\varphi}^-(x)\end{pmatrix}(\underline{\psi}^{+\mathrm{T}}(y),\quad \underline{\psi}^{-\mathrm{T}}(y)), \quad (2.5)$$

where the vector $\underline{\psi}^{\pm}(x)$ is $\underline{\psi}^{\pm}(x)=\underline{\varphi}^{\pm}(x)\boldsymbol{\mu}^{\mathrm{T}}$.

The matrix $\boldsymbol{I}-\tau\boldsymbol{J}\boldsymbol{\Phi}_n$ appearing under the sign of the Fredholm determinant can be represented as $\boldsymbol{I}+\boldsymbol{A}\boldsymbol{B}$ with

$$\boldsymbol{A}(x,\alpha)=-\tau\boldsymbol{J}\begin{pmatrix}\underline{\varphi}^+(x)\\ \underline{\varphi}^-(x)\end{pmatrix}=\tau\begin{pmatrix}-\underline{\varphi}^-(x)\\ \underline{\varphi}^+(x)\end{pmatrix}, \quad (2.6)$$

$$\boldsymbol{B}(\alpha,y)=(\underline{\psi}^{+\mathrm{T}}(y),\quad \underline{\psi}^{-\mathrm{T}}(y)). \quad (2.7)$$

Following Tracy and Widom (1998), one observes the 'needlessly fancy' general relation $\det[\boldsymbol{I} + \boldsymbol{A}\boldsymbol{B}] = \det[\boldsymbol{I} + \boldsymbol{B}\boldsymbol{A}]$ that holds for arbitrary Hilbert–Schmidt operators $\boldsymbol{A}$ and $\boldsymbol{B}$. They may act between different spaces as long as the products make sense. In the present context, $\det[\boldsymbol{I} + \boldsymbol{A}\boldsymbol{B}]$ is the Fredholm determinant $\det_X[\boldsymbol{I} + \boldsymbol{A}\boldsymbol{B}]$ whilst the determinant $\det[\boldsymbol{I} + \boldsymbol{B}\boldsymbol{A}]$ is that of the $n \times n$ matrix

$$\boldsymbol{I} + \tau \int_X m(\mathrm{d}x)(\underline{\psi}^{-\mathrm{T}}(x)\underline{\varphi}^{+}(x) - \underline{\psi}^{+\mathrm{T}}(x)\underline{\varphi}^{-}(x)) = \boldsymbol{I} + \tau\boldsymbol{\upsilon}, \tag{2.8}$$

where $\boldsymbol{\upsilon} = \boldsymbol{\mu}\boldsymbol{g}$ with $\boldsymbol{g}$ defined by equation (1.4).

The above calculation allows us to write down the Fredholm Pfaffian $\mathcal{S}(\tau; n)$ in the form

$$\mathcal{S}(\tau; n) = \sqrt{\det[\boldsymbol{I} + \tau\boldsymbol{\upsilon}]} = \exp\left[\frac{1}{2}\mathrm{tr}\log(\boldsymbol{I} + \tau\boldsymbol{\upsilon})\right] = \exp\left[\frac{1}{2}\sum_{j=1}^{\infty}(-1)^{j-1}\tau^j\frac{\mathrm{tr}(\boldsymbol{\upsilon}^j)}{j}\right]. \tag{2.9}$$

Identity equation (1.2) concludes the proof. □

3. Linear-algebraic proof of theorem 2

A linear-algebraic proof of theorem 2 is based on the works by Ishikawa and Wakayama (1995, 2000) and de Bruijn (1955) as formulated in sections 3.1 and 3.2. Section 3.3 contains a proof of theorem 2.

3.1. Minor summation formulae and identities by Ishikawa and Wakayama

Let $\boldsymbol{T}$ be any $M \times N$ matrix, and let $[m]$ denote the set $\{1, 2, \ldots, m\}$ for a positive integer $m \in \mathbb{Z}^+$. For n-element subsets $I = \{i_1 < \cdots < i_n\} \subseteq [M]$ and $J = \{j_1 < \cdots < j_n\} \subseteq [N]$ of row and column indices, let $\boldsymbol{T}^I_J = \boldsymbol{T}^{i_1\cdots i_n}_{j_1\cdots j_n}$ denote the submatrix of $\boldsymbol{T}$ obtained by picking up the rows and columns indexed by I and J. In this notation, the following three lemmas hold[7].

Lemma 1 (Ishikawa and Wakayama 1995)**.** *Let $M \leqslant N$ and assume M is even. For any $M \times N$ matrix $\boldsymbol{A}$ and any $N \times N$ antisymmetric matrix $\boldsymbol{B}$, one has*

$$\sum_{I\subseteq[N],\#I=M} \mathrm{pf}\big[\boldsymbol{B}^I_I\big]\det\big[\boldsymbol{A}^I_I\big] = \mathrm{pf}[\boldsymbol{A}\boldsymbol{B}\boldsymbol{A}^{\mathrm{T}}]. \tag{3.1}$$

Lemma 2 (Ishikawa and Wakayama 2000)**.** *Let $\boldsymbol{A}$ be an $N \times N$ invertible antisymmetric matrix. Then, for any $I \subseteq [N]$, one has*

$$\mathrm{pf}[\boldsymbol{A}]\mathrm{pf}\big[(\boldsymbol{A}^{-1})^I_I\big] = (-1)^{|I|}\mathrm{pf}\big[\boldsymbol{A}^{\bar{I}}_{\bar{I}}\big], \tag{3.2}$$

where $\bar{I} \subseteq [N]$ stands for the complementary of I, and $|I|$ denotes the sum of the elements of I, $|I| = \sum_{i\in I} i$.

Lemma 3 (Ishikawa and Wakayama 2000)**.** *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $N \times N$ antisymmetric matrices. Then*

$$\mathrm{pf}[\boldsymbol{A} + \boldsymbol{B}] = \sum_{r=0}^{\lfloor N/2\rfloor}\sum_{I\subseteq[N],\#I=2r}(-1)^{|I|-r}\mathrm{pf}\big[\boldsymbol{A}^I_I\big]\mathrm{pf}\big[\boldsymbol{B}^{\bar{I}}_{\bar{I}}\big]. \tag{3.3}$$

Here, $\lfloor n\rfloor$ denotes the integer part of n.

In what follows, we will need the following corollary.

[7] Lemma 2 is a reformulation of theorem 3.1 by Ishikawa and Wakayama (2000).

Corollary 1. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $N \times N$ antisymmetric matrices, and $\boldsymbol{A}$ is invertible. Then*

$$\sum_I \mathrm{pf}\big[\boldsymbol{A}_I^I\big]\mathrm{pf}\big[\boldsymbol{B}_I^I\big] = \mathrm{pf}[\boldsymbol{A}] \cdot \mathrm{pf}[\boldsymbol{A}^{-1\mathrm{T}} + \boldsymbol{B}]. \tag{3.4}$$

Proof. Using lemma 3 and lemma 2 (in this order), we write down

$$\begin{aligned}\mathrm{pf}[\boldsymbol{A}^{-1\mathrm{T}} + \boldsymbol{B}] &= \sum_{r=0}^{\lfloor N/2\rfloor} \sum_{I\subseteq[N],\#I=2r} (-1)^{|I|}\mathrm{pf}\big[(\boldsymbol{A}^{-1})_I^I\big]\mathrm{pf}\big[\boldsymbol{B}_{\bar{I}}^{\bar{I}}\big] \\ &= \frac{1}{\mathrm{pf}[\boldsymbol{A}]} \sum_{r=0}^{\lfloor N/2\rfloor} \sum_{I\subseteq[N],\#I=2r} \mathrm{pf}\big[\boldsymbol{A}_{\bar{I}}^{\bar{I}}\big]\mathrm{pf}\big[\boldsymbol{B}_{\bar{I}}^{\bar{I}}\big] \\ &= \frac{1}{\mathrm{pf}[\boldsymbol{A}]} \sum_I \mathrm{pf}\big[\boldsymbol{A}_I^I\big]\mathrm{pf}\big[\boldsymbol{B}_I^I\big].\end{aligned} \tag{3.5}$$

This concludes the proof. □

3.2. de Bruijn integration formula

Lemma 4 (de Bruijn 1955)**.** *Let (X, m) be a measure space, and the vectors $\underline{\varphi}^+(x)$ and $\underline{\varphi}^-(x)$ be composed of measurable functions $\underline{\varphi}^\pm(x) = \big(\varphi_0^\pm(x), \ldots, \varphi_{2\ell-1}^\pm(x)\big)$ from X to $\mathbb{C}$. Then*

$$\prod_{j=1}^{\ell} \int_X m(\mathrm{d}x_j)\, \det\begin{bmatrix}\varphi_j^+(x_i) \\ \varphi_j^-(x_i)\end{bmatrix}_{0\leqslant i\leqslant 2\ell-1, 1\leqslant j\leqslant\ell} = \ell!\,\mathrm{pf}\left[\boldsymbol{g}^{\mathrm{T}}\right],$$

where the $2\ell \times 2\ell$ matrix $\boldsymbol{g}$ is[8]

$$\boldsymbol{g} = \int_X m(\mathrm{d}x)(\underline{\varphi}^{-\mathrm{T}}(x)\underline{\varphi}^+(x) - \underline{\varphi}^{+\mathrm{T}}(x)\underline{\varphi}^-(x)).$$

3.3. Proof of theorem 2

In the notation of theorem 1, let us set $\boldsymbol{A}$ to be the $2\ell \times n$ matrix

$$\boldsymbol{A} = \begin{pmatrix}\varphi_j^+(x_i) \\ \varphi_j^-(x_i)\end{pmatrix}_{1\leqslant i\leqslant\ell, 0\leqslant j\leqslant n-1} \tag{3.6}$$

and identify $\boldsymbol{B}$ with the $n \times n$ matrix $\boldsymbol{\mu}$,

$$\boldsymbol{B} = \boldsymbol{\mu}. \tag{3.7}$$

Noting that

$$\mathrm{pf}[\boldsymbol{A}\boldsymbol{B}\boldsymbol{A}^{\mathrm{T}}] = \mathrm{pf}\begin{bmatrix}\Phi_n^{++}(x_i, x_j) & \Phi_n^{+-}(x_i, x_j) \\ \Phi_n^{-+}(x_i, x_j) & \Phi_n^{--}(x_i, x_j)\end{bmatrix}_{1\leqslant i,j\leqslant\ell}, \tag{3.8}$$

we make use of the lhs of lemma 1, to write down the expansion

$$\mathrm{pf}\begin{bmatrix}\Phi_n^{++}(x_i, x_j) & \Phi_n^{+-}(x_i, x_j) \\ \Phi_n^{-+}(x_i, x_j) & \Phi_n^{--}(x_i, x_j)\end{bmatrix}_{1\leqslant i,j\leqslant\ell} = \sum_{I\subseteq[n],\#I=2\ell} \mathrm{pf}\big[\boldsymbol{\mu}_I^I\big]\det\begin{bmatrix}\varphi_j^+(x_i) \\ \varphi_j^-(x_i)\end{bmatrix}_{1\leqslant i\leqslant\ell, j\in I}. \tag{3.9}$$

[8] Compare to equation (1.4).

Consequently, the lhs of equation (1.5) reduces to (note that n is even by the hypothesis)

$$\sum_{\ell=0}^{n/2}\frac{\tau^\ell}{\ell!}\sum_{I\subseteq[n],\#I=2\ell}\mathrm{pf}\big[\mu_I^I\big]\prod_{i=1}^{\ell}\int_X m(\mathrm{d}x_i)\det\begin{bmatrix}\varphi_j^+(x_i)\\ \varphi_j^-(x_i)\end{bmatrix}_{1\leqslant i\leqslant\ell,\, j\in I}$$
$$=\sum_{\ell=0}^{n/2}\tau^\ell\sum_{I\subseteq[n],\#I=2\ell}\mathrm{pf}\big[\mu_I^I\big]\mathrm{pf}\big[(g^{\mathrm{T}})_I^I\big]. \tag{3.10}$$

Here, we have used lemma 3. By corollary 1, the rhs of equation (3.10) is equivalent to

$$\mathrm{pf}[\mu]\cdot\mathrm{pf}[\mu^{-1\mathrm{T}}+\tau g^{\mathrm{T}}]=\mathrm{pf}[\mu]\cdot\mathrm{pf}[\mu^{-1\mathrm{T}}-\tau g]. \tag{3.11}$$

This concludes the proof. □

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Appendix. The Fredholm Pfaffian

Let $\boldsymbol{K}(x,y)$ be a 2×2 matrix kernel

$$\boldsymbol{K}(x,y)=\begin{pmatrix}K_{11}(x,y) & K_{12}(x,y)\\ K_{21}(x,y) & K_{22}(x,y)\end{pmatrix} \tag{A.1}$$

which is antisymmetric under the change of its arguments, $\boldsymbol{K}(x,y)=-\boldsymbol{K}^{\mathrm{T}}(y,x)$, and yet another 2×2 matrix kernel $\boldsymbol{J}(x,y)$ be

$$\boldsymbol{J}(x,y)=I(x,y)\tilde{\boldsymbol{J}},\qquad I(x,y)=\delta_{x,y},\qquad \tilde{\boldsymbol{J}}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}. \tag{A.2}$$

(i) The *Fredholm Pfaffian* $\mathrm{pf}_X[\boldsymbol{J}+\boldsymbol{K}]$ on the measure space (X,m) is defined via the series (Rains 2000)

$$\mathrm{pf}_X[\boldsymbol{J}+\boldsymbol{K}]=\sum_{\ell=0}^{\infty}\frac{1}{\ell!}\prod_{j=1}^{\ell}\int_X m(\mathrm{d}x_j)\mathrm{pf}[\boldsymbol{K}(x_i,x_j)]_{1\leqslant i,j\leqslant\ell}. \tag{A.3}$$

(ii) A more familiar *Fredholm determinant* $\det_X[I+K]$ of a scalar kernel K,

$$\det{}_X[I+K]=\sum_{\ell=0}^{\infty}\frac{1}{\ell!}\prod_{j=1}^{\ell}\int_X m(\mathrm{d}x_j)\det[K(x_i,x_j)]_{1\leqslant i,j\leqslant\ell}, \tag{A.4}$$

appears to be a particular case of the Fredholm Pfaffian since

$$\mathrm{pf}_X\left[\boldsymbol{J}+\begin{pmatrix}\varepsilon & K\\ -K & 0\end{pmatrix}\right]=\det{}_X[I+K]. \tag{A.5}$$

Here, ε is any antisymmetric scalar kernel.

(iii) The connection between the Fredholm Pfaffian and Fredholm determinant is given by

$$\mathrm{pf}_X[\boldsymbol{J}+\boldsymbol{K}]^2=\det{}_X[\boldsymbol{I}-\boldsymbol{J}\boldsymbol{K}], \tag{A.6}$$

where $\boldsymbol{I}$ is the 2×2 matrix kernel

$$\boldsymbol{I}(x,y)=I(x,y)\tilde{\boldsymbol{I}},\qquad \tilde{\boldsymbol{I}}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}. \tag{A.7}$$

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